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SIMPLE EXAMPLES OF NON LINEAR TRUSS BEHAVIOR†

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Abstract—Nonlinear material models are defined to represent elastic/perfectly-plastic and elastic/buckling behavior. A simple three-bar truss is used to demonstrate that under a monotonically increasing prescribed displacement the truss may exhibit reverse stressing or non-uniqueness, and that when two different control displacements are applied the principal of superposition does not hold.

1. INTRODUCTION

The theory of linear elasticity is a very "nice" mathematical theory with many convenient features such as superposition, uniqueness, and the equivalence of proportional loading to proportional stressing. Theories involving nonlinear material behavior do not necessarily have these nice features, and it is instructive to consider very simple examples which dramatically illustrate this fact.

In 1951 Drucker [1] considered a simple 3-bar truss essentially similar to the one shown in Fig. 1 which is subjected to a single monotonically increasing vertical load corresponding to the load Q. In Drucker's example each bar is made of an elastic/perfectly-plastic material with identical elastic properties but different yield strengths. For a suitable choice of yield strengths, bar 1 will first yield in compression but will then unload and will eventually yield in tension when the truss fails. The present author [2, 3] has made frequent use of trusses similar to the one in Fig. 1.

The present note shows how this truss can be used to discuss two different models of non-linear material behavior in relation to the above-mentioned features of superposition, uniqueness, and proportional stressing.

The two models to be treated are the familiar elastic/perfectly-plastic (E/PP) one illustrated by the dashed curve in Fig. 2, and an idealized elastic buckling (E/B) model indicated by the solid curve. Since we are primarily concerned with compressive stresses, we define the bar shortening and the negative bar force by

$$s_i^* = -L_i\epsilon_i \qquad C_i^* = -A_i\sigma_i. \tag{1}$$

The buckling model places no limit on negative values of C_i^* (it would be a trivial extension to construct a combined model which yielded in tension and buckled in compression) but an equally important difference in the two models occurs on unloading. At any point B, in the nonlinear range the E/PP model unloads along BCD whereas the E/B model retraces the loading path $BAOG$. This buckling model is slightly different than one used by Masur [4].

The next section lists the defining equations for the truss in Fig. 1 according to the two models, and the concluding section defines three specific examples chosen to illustrate reverse stressing under a monotonic load, non-uniqueness, and superposition, in that order. The results are summarized in Figs. 4–7 and Tables 1–4. Further details may be found in [5].

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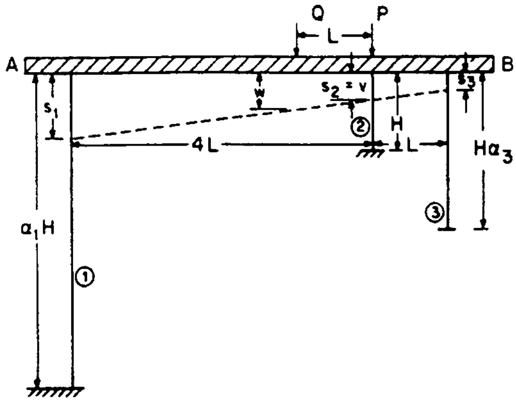

Fig. 1. Three-bar truss.

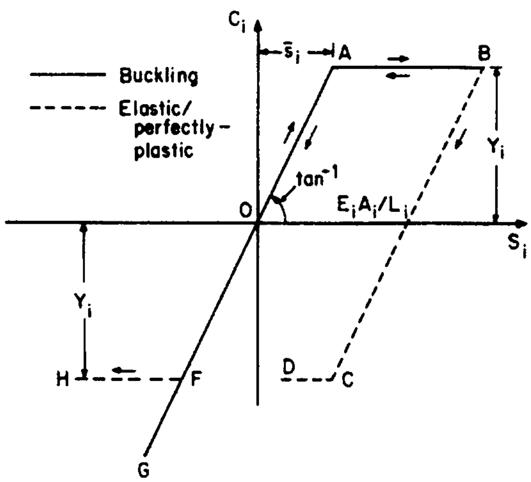

Fig. 2. Constitutive behavior of idealized models.

2. EQUATIONS

The cross bar AB is assumed rigid and the three vertical bars all have the same modulus E, area A, and minimum moment of inertia I. We find it convenient to use asterisks to represent physical quantities and define dimensionless variables as indicated below. For both models we define the stiffness of bar 2 by

$$k = AE/H \tag{2}$$

where H is the length of the bar. For the E/B model we use the simple Euler buckling formula for a pinned bar and denote the shortening of bar 2 at the onset of buckling by

$$\bar{s} = \frac{C_{cr}}{k} = \frac{\pi^2 EI}{H^2} \cdot \frac{H}{AE} = \frac{\pi^2 I}{AH}. \tag{3}$$

We then define dimensionless displacements v and w, loads P and Q, shortenings s_i, and compressive forces C_i by

$$\begin{aligned} v &= v^*/\bar{s} \qquad & w &= w^*/\bar{s} \qquad & s_i &= s_i^*/\bar{s} \\ P &= P^*/k\bar{s} \qquad & Q &= Q^*/k\bar{s} \qquad & C_i &= C_i^*/k\bar{s}. \end{aligned} \tag{4}$$

For the E/PP model, let Y be the yield force in bar 2 and define the above variables by the similar formulas

$$\begin{aligned} v &= kv^*/Y \quad & w &= kw^*/Y \quad & s_i &\ ks_i^*/Y \\ P &= P^*/Y \quad & Q &= Q^*/Y \quad & C_i &= C_i^*/Y. \end{aligned} \tag{5}$$

We denote the length of bar i by $\alpha_i H$ where $\alpha_2 = 1$ and we will choose α_1 and α_3 in each example. In view of the above definitions, the shortening at which bar i buckles is then given by

$$\bar{s}_i = 1/\alpha_i. \tag{6}$$

So that the two models will have the same scale in Fig. 2, we choose the dimensionless yield stress in bar i to be inversely proportional to the square of its length, hence

$$Y_i = 1/\alpha_i^2. \tag{7}$$

We assume that all displacements are small so that the only non-linearity is the material behavior, and the three bars can be taken to remain vertical. The kinematics are given by

$$\Delta s_i = (4\Delta w - 3\Delta v, \Delta v, 2\Delta v - \Delta w) \tag{8}$$

where we find it convenient to use the incremental form for all equations. The statics are obtained from moment equilibrium about the two loads:

$$\Delta P = -3\Delta C_1 + \Delta C_2 + 2\Delta C_3 \tag{9a}$$

$$\Delta Q = 4\Delta C_1 - \Delta C_3. \tag{9b}$$

Finally, the constitutive equation for each bar in the E/PP model may be written

$$\begin{gathered} \text{IF } (C_i = \pm 1/\alpha_i^2 \text{ AND } C_i\Delta s_i \geq 0) \\ \text{THEN } \Delta C_i = 0 \text{ ELSE } \Delta C_i = \Delta s_i/\alpha_i. \end{gathered} \tag{10}$$

For the E/B model the equation is similar but significantly different:

$$\begin{gathered} \text{IF } s_i > 1/\alpha_i \text{ THEN } \Delta C_i = 0 \\ \text{ELSE } \Delta C_i = \Delta s_i/\alpha_i. \end{gathered} \tag{11}$$

If each case the ELSE clause represents elastic behavior and the IF clause represents the idealized inelastic behavior. When all three bars are elastic, the solution for either model may be written

$$\Delta P = \left(\frac{9}{\alpha_1} + 1 + \frac{4}{\alpha_3}\right)\Delta v - \left(\frac{12}{\alpha_1} + \frac{2}{\alpha_3}\right)\Delta w \tag{12a}$$

$$\Delta Q = -\left(\frac{12}{\alpha_1} + \frac{2}{\alpha_3}\right)\Delta v + \left(\frac{16}{\alpha_1} + \frac{1}{\alpha_3}\right)\Delta w. \tag{12b}$$

Equations (12) could be solved for Δv and Δw. However, in each of our examples we will apply only one load at a time so that the zero load equation is trivially solved to relate Δv and Δw and the other equation relates load and displacement. In all cases, we will regard one of the displacements as the control variable.

Table 1. Unstressing. Control: $w(P = 0)$, $\alpha_i = (2, 1, 4)$

Line	Model	Stage	Status	w	v	Q	$16s_i$			$16C_i$		
1	E/PP	1L	EEE	1/4	1/4	7/16	4	4	4	2	4	1
2		2L	EEC	19/32	5/8	15/16	8	10	10.5	4	10	1
3		3L	CEE	51/32	7/8	17/16	60	14	2.5	4	14	-1
4		4L	CET	3	7/8	17/16	150	14	-20	4	14	-1
5	E/B	3L	BEB	1	5/8	15/16	34	10	4	4	10	1
6		4L	BEE	5/2	1	9/8	112	16	-8	4	16	-2
7		5L	BBE	3	5/4	9/8	132	20	-8	4	16	-2

3. EXAMPLES

As an example to illustrate unstressing we take $\alpha_i = (2, 1, 4)$, let Q be the only load, and increase the control displacement w under Q from 0 to 3. For the E/PP model this example is essentially similar to Drucker's [1].

The results are shown in Table 1 and Figs. 3 and 4. The "status" column in Table 1 shows for each bar if it is elastic (E), yielding in tension (T) or compression (C), or buckling (B). A "stage" is the time spent with no bar changing status, stage $1L$ is the limit of stage 1 as bar 1 reaches yield and changes from E to C, etc.

As shown in Table 1 as w is increased first bar 3 and then bar 1 yield in compression. However, as is clear from Fig. 1, a mechanism motion with bar 2 rigid would require bar 3 to lengthen. Therefore, in stage 3 bar 1 elastically unloads through zero and eventually yields in tension as the yield-point load is reached. The dashed curves in Figs. 3 and 4 respectively, show the load-displacement history and the shortening history of bar 3.

Stages 1 and 2 do not involve unstressing or tensile yield so that they are the same for the E/B model. However, in stage 3 as bar 3 starts to reclaim its buckling defor-

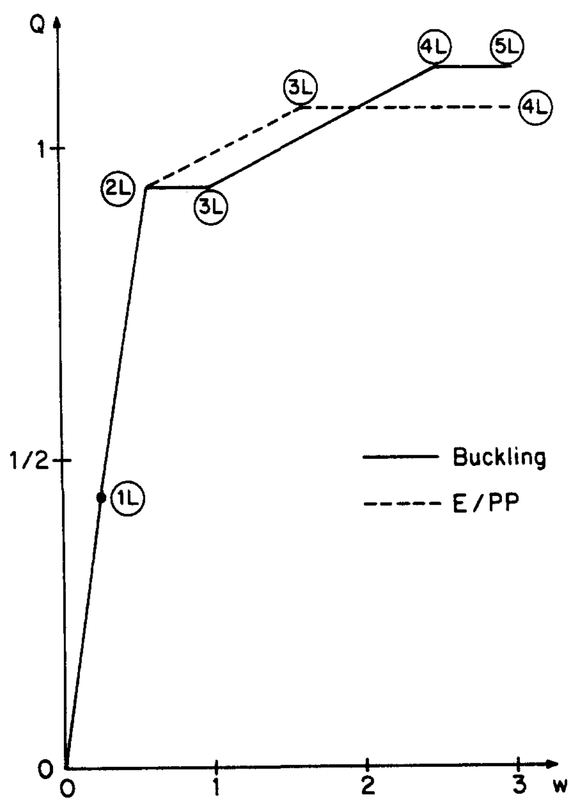

Fig. 3. Load-deformation curves for reversed stressing.

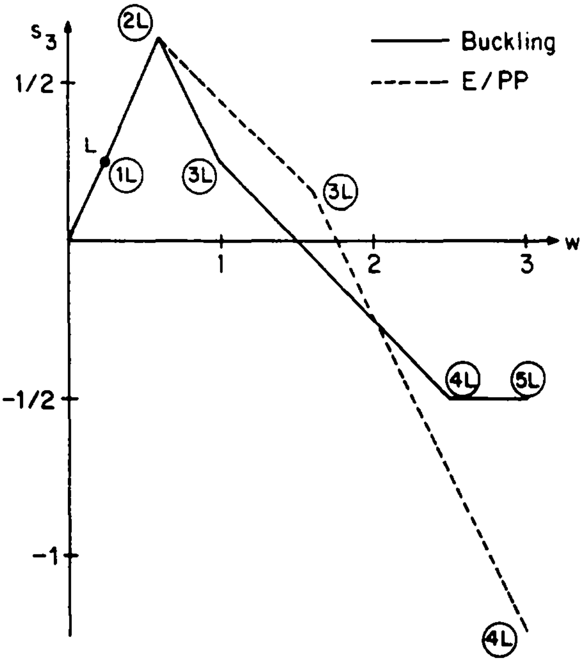

Fig. 4. Deformation of bar 3 for reversed stressing.

mation, there is no change in its force. Since bar 1 is now buckling C_1 also remains constant. Therefore, equilibrium shows that C_2 and the load are also constant. Thus, as w increases bar AB rotates about the unchanged location of the top of bar 2. However, this situation lasts only until s_3 is reduced to $\bar{s}_3$ at which point it resumes elastic behavior in stage 4. Since tensile yielding is not considered, stage 4 continues until bar 2 reaches buckling. The buckling collapse mechanism of the truss thus involves a rotation about the top of bar 1. The results are shown by the solid curves in Figs. 3 and 4. Notice the finite horizontal portion of the load-displacement curve in stage 3, followed by further increase of load in stage 4.

As a second example to illustrate non-uniqueness†, we consider a truss with $\alpha_i = (4, 1, 2)$ where P is the only load. The control variable v is to be increased from 0 to 1. For the E/PP model the all-elastic stage 1, shown in the top line of Table 2 ends when bars 1 and 3 both reach compressive yield at the same instant. Therefore, in stage 2, $\Delta C_1 = \Delta C_3 = 0$ and since there is no load Q eqn (9b) becomes an identity. Therefore, the only information about Δw comes from (8) and the inequalities in (10) for bars 1 and 3:

$$\Delta s_1 = 4\Delta w - 3\Delta v \geq 0 \qquad \Delta s_3 = 2\Delta v - \Delta w \geq 0. \tag{13}$$

These must apply for any infinitesimal increment in stage 2, which leads to

$$\tfrac{3}{4} \leq \mathrm{d}w/\mathrm{d}v \leq 2 \tag{14}$$

as shown in the last column for stage 2. During this stage the load P must increase with v. The bar forces are all unique, but w may take any value permitted by (14).

Table 2. Nonuniqueness. Control: $v(Q = 0)$; $\alpha_i = (4, 1, 2)$

Line	Model	Stage	Status	v	P	C_i			(w)
1	E/PP	1L	EEE	9/20	61/80	1/16	9/20	1/4	$w = 2/5$
2		2	CEC	Δv	Δv	0	Δv	0	$3/4 \leq dw/dv \leq 2$
3		2L	CCC	1	21/16	1/16	1	1/4	$13/16 \leq w \leq 3/2$

† Other simple examples of non-uniqueness are discussed in [6–7].

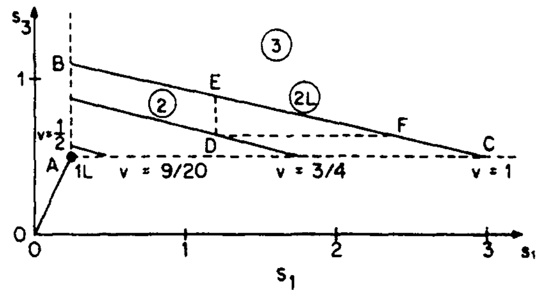

Fig. 5. Non-uniqueness of s_1 and s_3.

However, we note that once w has been established for any particular v, the restrictions (14) apply from that point. Therefore, although not unique as v is increased, any solution reached is stable if v is held constant at any time.

Figure 5 shows the strain-path trajectory of bars 1 and 3. It is unique along OA in stage 1, but during stage 2 it may follow any path with positive slope in the domain ABC. In particular, at stage $2L$ it may have reached any point on the line BC. However, if the solution at $v = \frac{3}{4}$ is observed to be at point D, say, then the possible solutions when $v = 1$ are restricted to the segment EF.

For the E/B model the equations and unique part of the results are exactly the same, so that we have not repeated them in Table 2. However, the inequalities apply to the total shortening rather than instantaneous increments, so that (13) and (14) must be replaced by

$$s_1 \geq \bar{s}_1 = \tfrac{1}{4} \qquad s_2 \geq \bar{s}_2 = \tfrac{1}{2}$$
$$\tfrac{3}{4}v + \tfrac{1}{16} \leq w \leq 2v - \tfrac{1}{2}. \tag{15}$$

Not only is the solution for w not unique, it is only neutrally stable and could change from one value to another with no change in v. Thus, at stage $2L$ it could be any point on BC regardless of its earlier values at $v = \frac{3}{4}$.

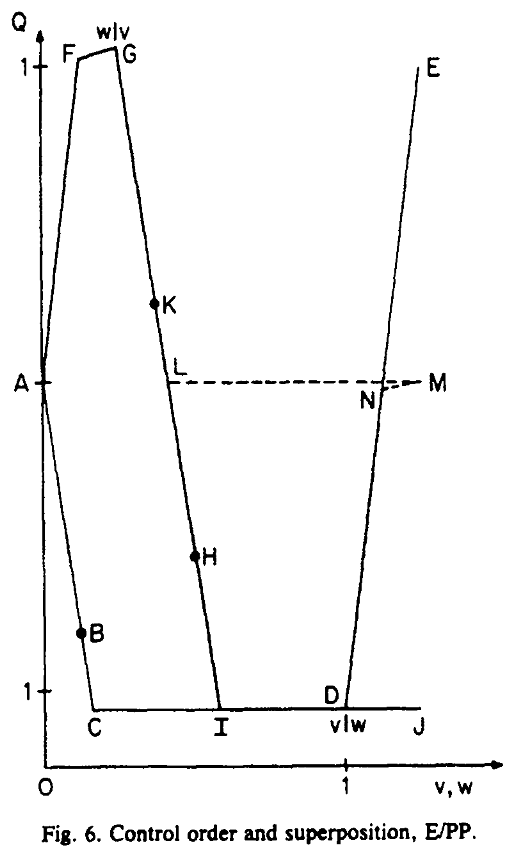

Fig. 6. Control order and superposition, E/PP.

Table 3. Superposition. Controls: v, w, α = (2, 1, 4)

Line	Model	History	w	v	Q	P	c_i		
1	E/PP	v then w	1/4	1	1	1/4	1/4	1	0
2		w then v	1/4	1	-17/16	15/8	-1/4	1	1/16
3		w = 0	0	1	-17/16	15/8	-1/4	1	1/16
4		v = 0	1/4	0	17/16	-7/8	1/4	0	-1/16
5		Superposition	1/4	1	0	1	0	1	0
6	E/B	Actual	1/4	1	-65/16	33/8	-1	1	1/16
7		w = 0	0	1	-97/16	45/8	-3/2	1	1/16
8		v = 0	1/4	0	17/16	-7/8	1/4	0	-1/16
9		Superposition	1/4	1	-5	19/8	-5/4	1	0

The final example discusses superposition. We return to the first truss where α_i = (2, 1, 4) and control both v and w, increasing them from zero to $w = \frac{1}{4}$, $v = 1$. For the E/PP model the final loads depend upon the order in which the displacements are increased. Suppose that they are applied in the order v, w; i.e. v is increased to 1 with w held at zero, then v is held at 1 while w is increased to $\frac{1}{4}$. The light solid curve *ABCDE* shows the history of load Q; P would have a similar curve. The final state when v = 1, $w = \frac{1}{4}$ is shown in line 1 Table 3.

On the other hand, if w is first increased to $\frac{1}{4}$ and then v is increased to 1, the history of Q is given by the heavy solid curve *AFGHIJ* in Fig. 6. Not only is the history quite different, but the final load values (points E and J) do not even agree in sign. Line 2 of Table 3 lists all final values.

If the principle of superposition were valid, we could find two components corresponding to v-only and w-only displacements and then add them. Lines 3–5 in Table 3 show the final results, and we see that line 5 is quite different than either lines 1 or 2. In terms of Fig. 6, we could add the w-only solution *AFG* to the v-only solution *ABCD* by translating the first curve so that A is at D. The resulting curve *ABCDNM* has its new part shown light dashed.

The order of superposition, of course, does not matter. If v-only is added to w-only, the curve *AFGKLM* has exactly the same terminal point M.

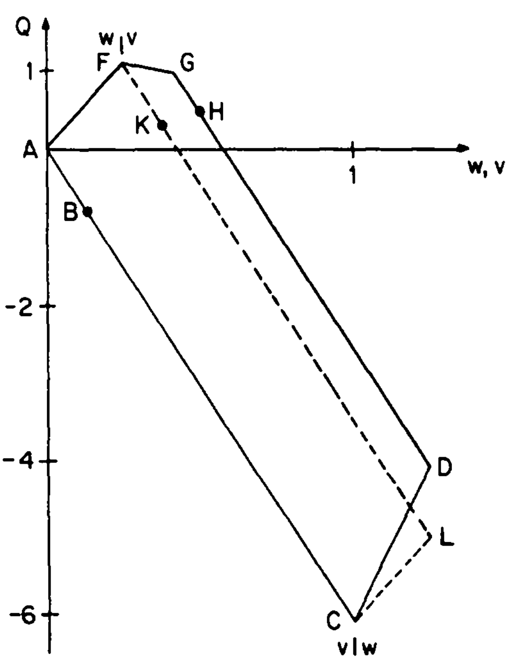

Fig. 7. Control order and superposition, E/B.

For the E/B model the shortenings, bar forces, and loads are all unique functions of the instantaneous displacements, so that curves *ABCD* and *AFGHD* in Fig. 7 both end at the same state *D*. The complete solution at the final point $v = 1$, $w = \frac{1}{4}$ is shown in line 6 of Table 3. Observe that the loads are very much different from those required by the E/PP model, essentially because the latter had bar 1 with substantial yielding in tension.

However, as shown by lines 7–9 in Table 3 and by curves *ABCL* or *AFKL* in Fig. 7, the results of superposition still do not agree with the actual solution.

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